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A Viterbi Algorithm for Modulo-2π Phase Tracking

in

Coherent Data Communication Systems

by

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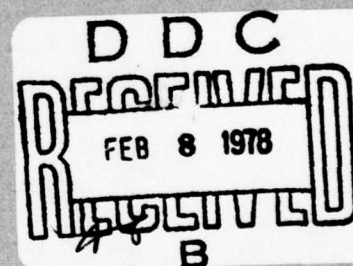by

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Abstract

It is argued that only phase, modulo-2π, is of interest in data communication applications. The probabilistic evolution of random walk phase, modulo-2π, is studied and the results used to derive a MAP sequence estimator for phase on the interval $[-\pi,\pi)$. The MAP sequence estimator is a Viterbi tracker (or dynamic programming algorithm) for tracking phase on a finite-dimensional trellis in $[-\pi,\pi)$. Computational requirements are discussed and several representative simulations are included. Applications of the algorithm to carrier phase tracking in coherent data communication systems are outlined.

I. Introduction

A general model for data transmission over an imperfect bandlimited channel includes the effects of intersymbol interference, carrier phase fluctuation, and additive receiver noise. The carrier phase fluctuation arises from changes in the channel transmission characteristics, ripple in dc power supplies, and timing jitter at the transmitter and receiver. A somewhat primitive, but tractable, model for carrier phase fluctuation is the so-called random walk. A more sophisticated model would account for the fundamental and harmonic components of ringing currents and 60Hz interference [1].

In this paper we propose a new approach to carrier phase tracking. We observe that the measured signal in a data communication system is invariant to modulo-2π transformations on the carrier phase. Therefore maximum *a posteriori* (MAP) sequence estimation for a modulo-2π version of the underlying phase sequence seems an appropriate estimation problem to pose. To solve this sequence estimation problem we derive an expression for transition probabilities in a modulo-2π version of random walk. A rigorous treatment may be given by studying random walk on the circle [2]. The modulo-2π transition probabilities are used to derive a path metric for use in a Viterbi algorithm for tracking phase, modulo-2π, on a finite-dimensional trellis in $[-\pi,\pi)$. The resulting algorithm approximates the maximum *a posteriori* (MAP) modulo-2π phase sequence. Several representative simulations are presented to illustrate the tracking characteristics of the proposed algorithm. Applications of the algorithm are outlined for phase coherent data communication over channels that may or may not have been equalized.

II. Signal and Phase Models

We begin our development by considering the output of a quadrature demodulator when there is no data transmission taking place. This allows us to formulate and solve the modulo-2π carrier phase tracking problem. The solution is generalized in Section VIII to include simultaneous carrier phase tracking and symbol detection.

Consider the following complex representation for the sample values appearing at the output of a complex demodulator:

$$z_k = \exp(j\phi_k) + n_k \quad , \quad k=0,\pm1,\pm2, \ldots \tag{1}$$

The additive noise sequence $\{n_k\}$ is assumed to be a sequence of independent identically-distributed (i.i.d.) complex normal random variables:

$$n_k = u_k + jv_k$$
$$u_k : N(0,\sigma^2) \qquad v_k : N(0,\sigma^2) \tag{2}$$
$$u_k \perp\!\!\!\perp v_k$$

The notation $x:N(\mu,\sigma^2)$ indicates that x is a real-valued normal random variable with mean μ and variance σ^2. The notation $x \perp\!\!\!\perp y$ means x and y are statistically independent.

The random phase sequence $\{\phi_k\}$ appearing in (1) is assumed to be a random walk sequence characterized by i.i.d. normal increments w_k:

$$\phi_{k+1} = \phi_k + w_k$$
$$w_k : N(0,\sigma_w^2) \qquad w_k \perp\!\!\!\perp w_\ell \;\forall\; k \neq \ell \tag{3}$$

Let $f(x/y)$ denote the conditional density function for the random variable x, conditioned on (a realization of) y.[1] Then the conditional

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[1] What we really mean here is the conditional density function for a random variable x, given that y equals some value α. A more appropriate, but also more cumbersome, notation would be $f_{x/y}(\cdot/\alpha)$.

density for ϕ_{k+1}, given ϕ_k, is

$$f(\phi_{k+1}/\phi_k) : N(\phi_k, \sigma_w^2) \tag{4}$$

By this notation we mean "... conditioned on the value ϕ_k for the phase at time k, the phase at time k+1 is a normal random variable with mean ϕ_k and variance σ_w^2."

Note that the measurement model of (1) is invariant to a modulo-2π transformation of the phase ϕ_k. That is

$$z_k = \exp(j\bar{\phi}_k) + n_k \tag{5}$$

where $\bar{\phi}_k$ is the modulo-2π version of ϕ_k:

$$\phi_k = \bar{\phi}_k + \ell 2\pi \quad \text{some } \ell \quad ; \quad \bar{\phi}_k \in [-\pi, \pi) \tag{6}$$

This observation motivates us to model the probabilistic evolution

of the phase sequence $\{\bar{\phi}_k\}$ and derive a MAP sequence estimator for the

modulo-2π sequence $\{\bar{\phi}_k\}$.

III. Review of Ungerboeck [1] and Magee [3]

Ungerboeck [1] and Magee [3] have treated carrier phase tracking

in the context of simultaneous phase estimation and symbol detection.

Ungerboeck assumed a perfectly equalized channel and Magee allowed for

a finite-length channel impulse response. For purposes of comparison

with the approach proposed in this paper, we present a brief review

of [1] and [3].

In his treatment of phase sequence estimation Ungerboeck observed

that the phase ϕ_k is a continuous-valued variable and that no Viterbi-

like algorithm was directly applicable. He, therefore, discretized the

range space for ϕ_k to a grid of spacing σ_w (in our notation). He called

this approximating sequence a sequence with discrete binary increments

and the corresponding approximation a <u>delta-mod approximation.</u> The

underlying parameter space is now countably, rather than noncountably, infinite. Ungerboeck handles this difficulty by restricting attention to only those phase states that lie in a region around the developing most-likely path. Magee does not really implement a phase _sequence_ estimator. Rather, in the interest of conserving computation, he first minimizes a path metric with respect to the current phase value (at each step in the recursion) and then uses a dynamic programming algorithm to find a MAP data sequence estimate.

IV. The MAP Sequence Estimation Problem for Modulo-2π Phase

Consider the following maximization problem with respect to the modulo-2π phase sequence $\{\bar{\phi}_k\}_1^K$:

$$\max_{\{\bar{\phi}_k\}_1^K} f(\{z_k\}_1^K, \{\bar{\phi}_k\}_1^K) \tag{7}$$

Here $f(\cdot,\cdot)$ is the joint density function for the K measurements $\{z_k\}_1^K$ and the K modulo-2π phase values $\{\bar{\phi}_k\}_1^K$. Maximization of this joint density function is equivalent to maximization of the a posteriori density $f(\{\bar{\phi}_k\}_1^K/$ $\{z_k\}_1^K)$.

Write the joint density of (7) as

$$f(\{z_k\}_1^K, \{\bar{\phi}_k\}_1^K) = f(\{z_k\}_1^K/\{\bar{\phi}_k\}_1^K) f(\{\bar{\phi}_k\}_1^K)$$

$$= f(\{\bar{\phi}_k\}_1^K) \prod_{k=1}^K N_{z_k}(e^{j\bar{\phi}_k}, \sigma^2) \tag{8}$$

The last simplification in (8) follows from the fact that the n_k in (1) are i.i.d. normal random variables and the fact that conditionally,

$$f(z_k/\bar{\phi}_k) : N_{z_k}(e^{j\bar{\phi}_k}, \sigma^2)$$

We have used the subscripted notation $N_{z_k}(e^{j\bar{\phi}_k}, \sigma^2)$ to indicate that

the conditional density of the complex random variable z_k (conditioned on $\bar{\phi}_k$) is normal with mean $e^{j\bar{\phi}_k}$ and variance σ^2:

$$N_{z_k}(e^{j\bar{\phi}_k}, \sigma^2) = (2\pi\sigma^2)^{-1} \exp\{-\frac{1}{2\sigma^2} |z_k - e^{j\bar{\phi}_k}|^2\} \tag{9}$$

Dropping phase independent terms we may write the MAP sequence estimation problem as

$$\max_{\{\bar{\phi}_k\}_1^K} [\ell nf(\{\bar{\phi}_k\}_1^K) + \frac{1}{2\sigma^2} 2Re \sum_{k=1}^K z_k e^{-j\bar{\phi}_k}] \tag{10}$$

Or, defining the envelope statistic $C_k = |z_k|$ and the phase statistic $\psi_k = \arg z_k$ (so that $z_k = |z_k| \exp(j\psi_k)$), we may write

$$\max_{\{\bar{\phi}_k\}_1^K} [\ell nf(\{\bar{\phi}_k\}_1^K) + \frac{1}{2\sigma^2} 2 \sum_{k=1}^K C_k \cos(\psi_k - \phi_k)] \tag{11}$$

The function $f(\{\bar{\phi}_k\}_1^K)$ is the joint density function of the K modulo-2π phase values $\{\bar{\phi}_k\}_1^K$. We study this object in the next section. We emphasize that $\psi_k = \arg z_k$ takes on values only in $[-\pi,\pi)$.

V. Probabilistic Evolution of the Modulo-2π Phase Sequence

We are interested in the probabilistic evolution of the sequence $\{\bar{\phi}_k\}$. Accordingly, we seek an expression for the conditional density

$$f(\bar{\phi}_{k+1}/\bar{\phi}_k) \tag{12}$$

This will permit us to write

$$f(\{\bar{\phi}_k\}_1^K) = \prod_{k=1}^K f(\bar{\phi}_k/\bar{\phi}_{k-1}) ; f(\bar{\phi}_1/\bar{\phi}_0)\underline{\Delta}f(\bar{\phi}_1) \tag{13}$$

because the modulo-2π transformation does not destroy the Markov property property in a random walk.

Note $\bar{\phi}_{k+1}$ may be written

$$\bar{\phi}_{k+1} = \bar{\bar{\phi}}_{k+1} ; \bar{\bar{\phi}}_{k+1} = \bar{\phi}_k + w_k \tag{14}$$

The conditional density of $\tilde{\phi}_{k+1}$ is

$$f(\tilde{\phi}_{k+1}/\bar{\phi}_k) \quad : \quad N(\bar{\phi}_k, \sigma_w^2) \tag{15}$$

Since $\bar{\phi}_{k+1}$ is a modulo-2π version of $\tilde{\phi}_{k+1}$ we may simply reflect all of the conditional probability mass into $[-\pi, \pi)$ to obtain the result

$$f(\bar{\phi}_{k+1}/\bar{\phi}_k) = \sum_{n=-\infty}^{\infty} N(\bar{\phi}_k + n2\pi, \sigma_w^2) \tag{16}$$

The importance of this result is the following: because of the Markov property of $\{\bar{\phi}_k\}$ we may write the joint density of (13) with the conditional density evaluated as in (16). And as we will find in Section VI, (16) may be used to assign transition probability mass to a finite grid of phase values. These transition probabilities then enter into the path metric in a Viterbi algorithm for tracking modulo-2π phase.

There is an important symmetry property of (16): the transition probability depends only on the squared distance between $\bar{\phi}_k$ and $\bar{\phi}_{k+1}$. Thus if (16) is evaluated on an LxL grid of phase pairs, there is Toeplitz symmetry in the corresponding matrix of transition probabilities. This means only an L-vector of transition probabilities need be stored for cyclic reading to obtain the entries in the matrix.

VI. The Viterbi Tracker for Modulo-2π Phase

Using the result of (13), write out the maximization problem of (11) as follows:

$$\max_{\{\bar{\phi}_k\}_1^K} \Gamma_K$$

$$\Gamma_K = \sum_{k=1}^{K} \ln f(\bar{\phi}_k/\bar{\phi}_{k-1}) + \frac{1}{2\sigma^2} 2 \sum_{k=1}^{K} C_k \cos(\psi_k - \bar{\phi}_k) \tag{17}$$

Note that Γ_k satisfies the recursion

$$\Gamma_k = \Gamma_{k-1} + \ln f(\bar{\phi}_k/\bar{\phi}_{k-1}) + \frac{1}{2\sigma^2} 2C_k \cos(\psi_k - \bar{\phi}_k)$$

$$\Gamma_1 = \ln f(\bar{\phi}_1) + \frac{1}{2\sigma^2} 2C_1 \cos(\psi_1 - \bar{\phi}_1) \tag{18}$$

Thus the so-called path metric is

$$\ln f(\bar{\phi}_k/\bar{\phi}_{k-1}) + \frac{1}{2\sigma^2} 2C_k \cos(\psi_k - \bar{\phi}_k) \tag{19}$$

The maximization problem may now be written

$$\max_{\{\bar{\phi}_k\}_{K-1}^K} [\max_{\{\bar{\phi}_k\}_1^{K-2}} \Gamma_{K-1} + \ln f(\bar{\phi}_K/\bar{\phi}_{K-1}) + \frac{1}{2\sigma^2} 2C_K \cos(\psi_K - \bar{\phi}_K)] \tag{20}$$

This form leads to the following observation: the maximizing trajectory (call it $\{\hat{\bar{\phi}}_k\}_1^K$), passing through $\hat{\bar{\phi}}_{K-1}$ on its way to $\hat{\bar{\phi}}_K$, must arrive at $\hat{\bar{\phi}}_{K-1}$ along a route $\{\hat{\bar{\phi}}_k\}_1^{K-2}$ that maximizes Γ_{K-1}. If it did not we could retain $\hat{\bar{\phi}}_{K-1}$ and $\hat{\bar{\phi}}_K$ and replace $\{\hat{\bar{\phi}}_k\}_1^{K-2}$ with a different sequence to get a larger value for Γ_K. It is this observation which forms the basis of forward dynamic problem. The Viterbi algorithm, a forward dynamic programming algorithm applied to MAP sequence estimation on finite-state sequences, may be applied.

The trellis of Fig. 1 illustrates how the minimization of (20) proceeds. Tabulated values of $f(\bar{\phi}_k/\bar{\phi}_{k-1})$ are stored in square array
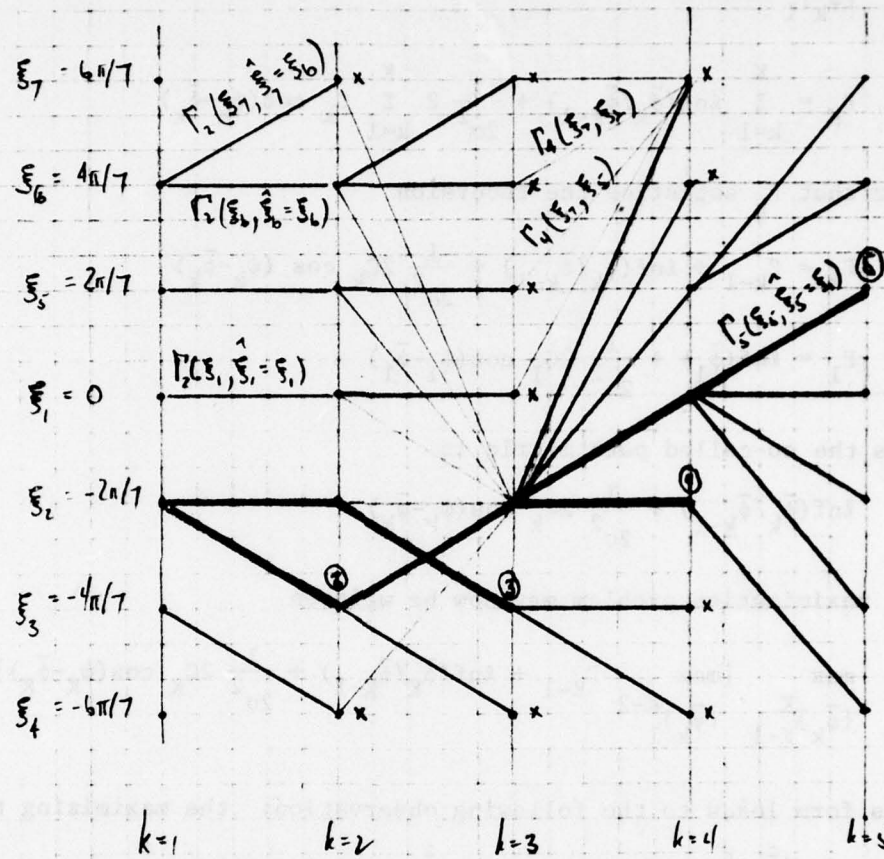
$\xi_7 = 6\pi/7$

$\xi_6 = 4\pi/7$

$\xi_5 = 2\pi/7$

$\xi_1 = 0$

$\xi_2 = -2\pi/7$

$\xi_3 = -4\pi/7$

$\xi_4 = -6\pi/7$

$\Gamma_2(\xi_7, \hat{\xi}_7 = \xi_6)$

$\Gamma_2(\xi_6, \hat{\xi}_6 = \xi_6)$

$\Gamma_4(\xi_7, \xi_7)$

$\Gamma_4(\xi_7, \xi_6)$

$\Gamma_5(\xi_5, \hat{\xi}_5 = \xi_6)$

$\Gamma_2(\xi_1, \hat{\xi}_1 = \xi_1)$

$k=1 \qquad k=2 \qquad k=3 \qquad k=4 \qquad k=5$

Fig. 1. Phase Trellis Illustrating the Evolution of Surviving Phase Tracks

(or vector which is cyclically read) whose dimensions depend upon how finely the interval $[-\pi,\pi)$ is discretized. Call $\bar{=} \times \bar{=}$, with $\bar{=} = \{\S_\ell\}_1^L$, the finite-dimensional grid for which $f(\bar\phi_k/\bar\phi_{k-1})$ is defined. That is, $\bar\phi_k$ is assumed to take on only the values $\bar\phi_k = \S_\ell$, $\ell=1,2,\ldots,L$, for each k. Call $\Gamma_k(\S_\ell,\S_m)$ the value of the metric Γ_k corresponding to a phase trajectory $\{\bar\phi_j\}_1^k$ which terminates at phase-state $\S_\ell$ at stage k, after passing through stage $\S_m$ at stage k-1.

The algorithm begins with a computation of $\Gamma_1(\S_\ell,\S_\ell)$, $\ell=1,2,\ldots,L$ based on measured values of C_1 and ψ_1. See (18). If all phase values are equally likely <u>a priori</u>, then $\ell n f(\bar\phi_1)$ is constant on all values $\S_\ell$. Otherwise there is some <u>a priori</u> weighting in favor of some of the $\S_\ell$. A new measurement pair (C_2,ψ_2) is obtained at k=2 and $\Gamma_2(\S_1,\S_m)$ is computed for $m=1,2,\ldots,L$ using a table look-up (e.g. in ROM) for the $f(\bar\phi_2=\S_1/\bar\phi_1=\S_m)$. The maximum value of $\Gamma_2(\S_1,\S_m)$ is determined (the maximization is over all originating states $\S_m$) and the corresponding sequence $(\S_1,\hat\S_1)$ is saved as a *survivor* sequence terminating at $\S_1$ at stage 2; $\hat\S_1$ denotes the originating state. The survivor sequence is labelled with its corresponding *length* $\Gamma_2(\S_1,\hat\S_1)$. This calculation is repeated for each possible value of phase until all pairs $(\S_\ell,\hat\S_\ell)$, and corresponding lengths $\Gamma_2(\S_\ell,\hat\S_\ell)$, $\ell=1,2,\ldots,L$, have been computed and stored in soft memory (e.g. RAM). There is a unique survivor sequence corresponding to each state $\S_\ell$, $\ell=1,2,\ldots,L$. *Caution:* In the pair $(\S_\ell,\hat\S_\ell)$ the originating state $\hat\S_\ell$ depends on $\S_\ell$; i.e., $\hat\S_\ell = \hat\S_\ell(\S_\ell)$. The measurements C_2 and ψ_2 may be discarded along with all extinct sequences. Now a new measurement pair (C_3,ψ_3) is obtained and $\Gamma_3(\S_1,\S_m)$ is computed for $m=1,2,\ldots,L$ using the recursion for Γ_k and a table look-up for $f(\bar\phi_3=\S_1/\bar\phi_2=\S_m)$. The minimizing value of $\Gamma_3(\S_1,\S_\ell)$, call it $\Gamma_3(\S_1,\hat\S_1)$

is computed and the corresponding triple $(\S_1, \hat{\S}_1, \cdot)$ is stored as a survivor sequence originating in $(\cdot)$ at stage 1, passing through $\hat{\S}_1$ at stage 2, and terminating in $\S_1$ at stage 3. This procedure is repeated until L survivor sequences of the form $(\S_\ell, \hat{\S}_\ell, \cdot)$ are determined and stored with their respective survivor metrics $\Gamma_3(\S_\ell, \hat{\S}_\ell \cdot)$. The measurements (C_3, ψ_3) and all extinct sequences may be discarded.

Call $(\hat{\bar{\phi}}_1(k), \hat{\bar{\phi}}_2(k), \ldots, \hat{\bar{\phi}}_k(k))$ the MAP sequence estimate based on k measurements up to stage k. This sequence has the maximum value of Γ_k. The parenthetical notation (k) denotes dependence on measurement interval. In general the MAP sequence estimate, $(\hat{\bar{\phi}}_1(k+1),$ $\ldots, \hat{\bar{\phi}}_{k+1}(k+1))$, based on measurements up to stage k+1, may differ from the previous sequence estimate at every stage from 1 to k. However, as a practical matter, one can choose a sufficiently large *depth* parameter k_0 so that the sequence of fixed-lag estimates

$$\hat{\bar{\phi}}_{k-k_0}(k) \quad , \quad k = k_0, \ k_0+1, \ldots \tag{21}$$

gives an approximate MAP sequence estimate. Here $\hat{\bar{\phi}}_{k-k_0}(k)$ is simply the phase value, k_0 stages back, in the MAP sequence estimate based on k measurements. In this way one obtains a phase track with delay k_0.

Following Forney [4] we may summarize the storage and computational requirements for the phase tracking algorithm as follows:

Storage

k	(time index)
$(\S_\ell, \hat{\S}_\ell, \cdot, \ldots), \ell=1,2,\ldots,L$	(survivor phase sequence terminating in $\S_\ell$ at stage k)
$\Gamma_k(\S_\ell, \hat{\S}_\ell, \ldots), \ell=1,2,\ldots,L$	(survivor metric)
$f(\S_\ell / \S_m), \ \ell=1,\ldots,L$ $m=1,\ldots L$	(transition probability matrix)

Initialization

$$k=1$$

$$\hat{\S}_\ell = \S_\ell \ , \quad \ell=1,\ldots,L$$

$$\Gamma_1(\S_\ell,\hat{\S}_\ell) = \ell n f(\bar{\phi}_1=\S_\ell) + \frac{1}{2\sigma^2} 2C_1 \cos(\psi_1-\S_\ell), \ell=1,2,\ldots,L$$

Recursion

$$\Gamma_{k+1}(\S_\ell,\S_m) = \Gamma_k(\S_m,\hat{\S}_m) + \ell n f(\bar{\phi}_{k+1}=\S_\ell/\bar{\phi}_k=\S_m)$$

$$+ \frac{1}{2\sigma^2} 2C_{k+1} \cos(\psi_{k+1}-\S_\ell) \ , \quad \ell=1,2,\ldots,L$$

$$\min_{\{\S_m\}} \Gamma_{k+1}(\S_\ell,\S_m) \ , \quad \ell=1,2,\ldots,L$$

Measurement/Computation

$$C_k \qquad\qquad \text{envelope statistic}$$

$$\psi_k \qquad\qquad \text{phase statistic}$$

$$C_k \cos(\S_\ell-\psi_k) + \ell n f(\bar{\phi}_k=\S_\ell/\bar{\phi}_{k-1}=\S_m) \quad \text{path metric}$$

In Fig. 1 typical trajectories for this algorithm are illustrated. The heavy lines denote survivors and the light lines denote path metric calculations that are made and then discarded in favor of survivors. At stage 3 all calculations $\Gamma_3(\S_2,\S_m)$ are illustrated with light lines; the heavy line from $\S_3$ to $\S_2$ illustrates that this path gives maximum $\Gamma_2(\S_2,\S_m)$ and is therefore labelled a survivor. Of course $\hat{\S}_2 = \S_3$. The x's on the trellis illustrate sequences that have survived for a while before being exterminated by the weight of evidence. The very heavy line at each measurement stage k denotes the current MAP sequence. The circled number denotes the current MAP sequence. The sequence of end points labelled with the circled numbers is a sequence of phase estimates. Note this sequence of phase estimates differs

from the <u>MAP phase sequence estimate.</u> The latter, being a smoothing

solution, is in fact, generally smoother than the former.

VII. Simulations

In the simulations described here the modulo-2π phase space $[-\pi,\pi)$

has been discretized to M points $\S_\ell = \ell 2\pi/L - (L-1)\pi/L$, $\ell=0$, 1,... (L-1). In

Figs. 2 and 3, L=11. In Fig. 4, L=17. The transition probabilities

between these phase points have been computed by approximating (16) on

an LxL grid. These values have been stored in an LxL matrix (really an

L-vector which is cyclically permuted to generate the rows of the Toeplitz

transition matrix) and used to calculate path metrics according to (19).

A random walk phase sequence $\{\phi_k\}$ was generated from (3) and used

to construct the sequence of measurements given in (1). These measure-

ments were then used in the dynamic programming algorithm of Section VI

to generate the results of Figs. 2-4.

Fig. 2 is a relatively high signal-noise ratio example where

$\sigma_n^2 = 0.1$ (S/N = 10dB). The phase variance parameter is $\sigma_w^2 = (\pi/10\sqrt{2})^2$,

corresponding to a phase trajectory whose "1-σ transitions" are on the

order of $\pi/14$. In Fig. 2 the solid curve with piecewise constant slopes

is the actual phase sequence $\{\phi_k\}$. The dotted curve with piecewise

constant slopes is the measurement sequence $\{\psi_k\}$, with ψ_k = arg z_k

$\epsilon[-\pi,\pi)$. The heavy lines labelled with numbers from 1 to 37 are MAP

phase sequences generated with the modulo-2π Viterbi algorithm of

Section VI. The MAP sequence after k measurements is labelled with the

corresponding k value at the end of the sequence. The sample rms phase

error is approximately $\pi/10$, which is about twice the rms quantization
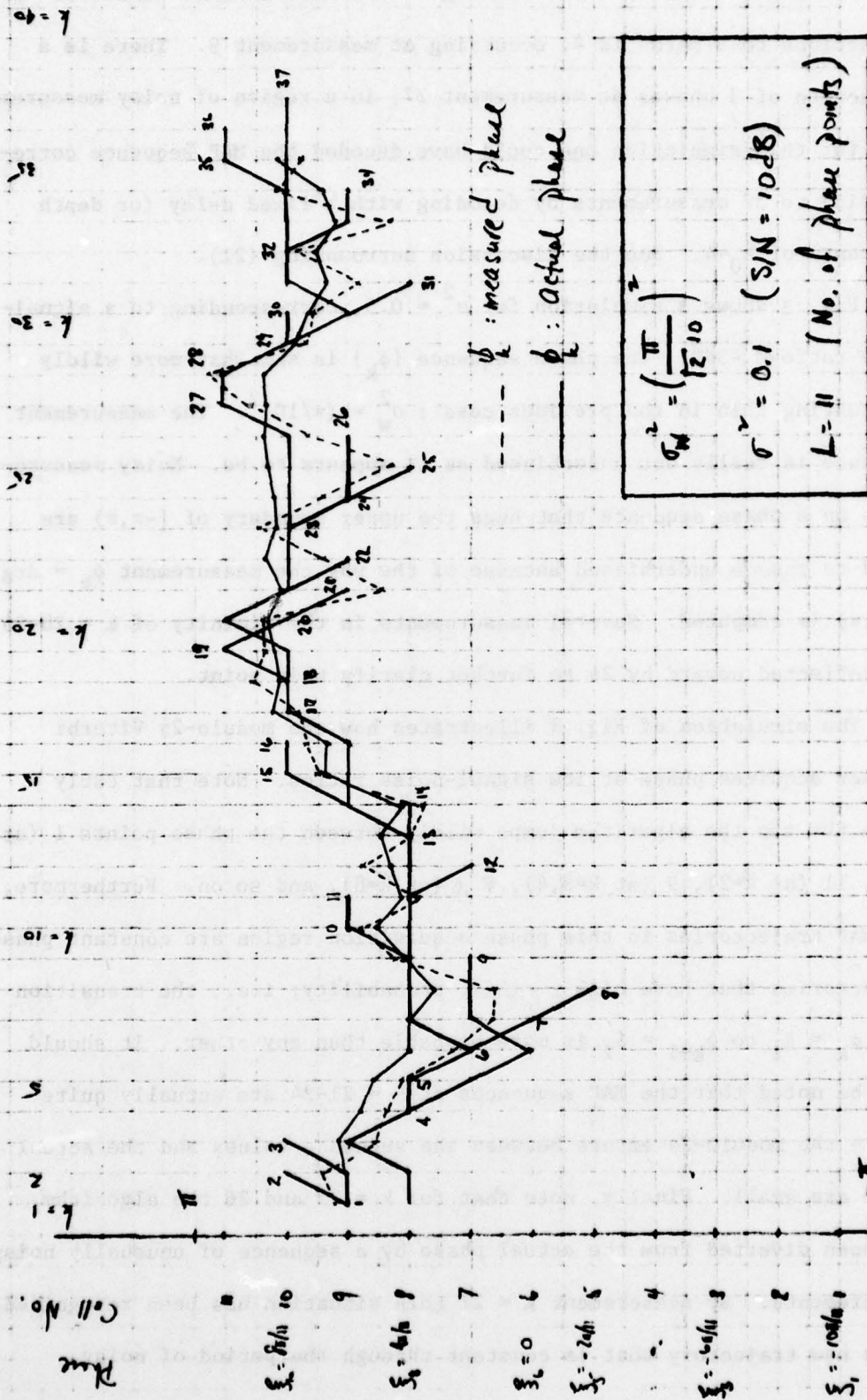
Fig. 2. Typical Modulo–2π MAP Phase Sequences

noise level of $(\pi/11\sqrt{3})$. Note in Fig. 2 that the maximum number of "phase corrections to a path" is 4, occurring at measurement 9. There is a correction of 3 phases at measurement 27, in a region of noisy measurements. Thus for this simulation one could have decoded the MAP sequence corresponding to 37 measurements by decoding with a fixed delay (or depth constant) of $k_0=4$. See the discussion surrounding (21).

Fig. 3 shows a simulation for $\sigma^2 = 0.5$, corresponding to a signal-noise ratio of -3dB. The phase sequence $\{\phi_k\}$ is somewhat more wildly fluctuating than in the previous case : $\sigma_w^2 = (\pi/10)^2$. The measurement sequence is really not underbiased as it appears to be. Noisy measurements on a phase sequence that hugs the upper boundary of $[-\pi,\pi)$ are bound to *appear* underbiased because of the way the measurement $\psi_k = \arg z_k$ $\epsilon[-\pi,\pi)$ is computed. Several measurements in the vicinity of $k = 28-39$ are reflected upward by 2π to further clarify this point.

The simulation of Fig. 3 illustrates how the modulo-2π Viterbi tracker acquires phase at low signal-noise ratios. Note that early on in the run the algorithm jumps wildly between the phase points 1 (at $k=1$), 11 (at $k=2$), 9 (at $k=3,4$),...,6 (at $k=8$), and so on. Furthermore, the MAP trajectories in this phase acquisition region are constant phase trajectories that have high *a priori* probability; i.e., the transition from $\phi_k = \S_\ell$ to $\phi_{k+1} = \S_\ell$ is more probable than any other. It should also be noted that the MAP sequences at $k = 21-24$ are actually quite good – the modulo-2π errors between the sequence values and the actual phase are small. Finally, note that for $k = 25$ and 26 the algorithm has been diverted from the actual phase by a sequence of unusually noisy measurements. By measurement $k = 27$ this situation has been recognized and a new trajectory that is constant through the period of noisy
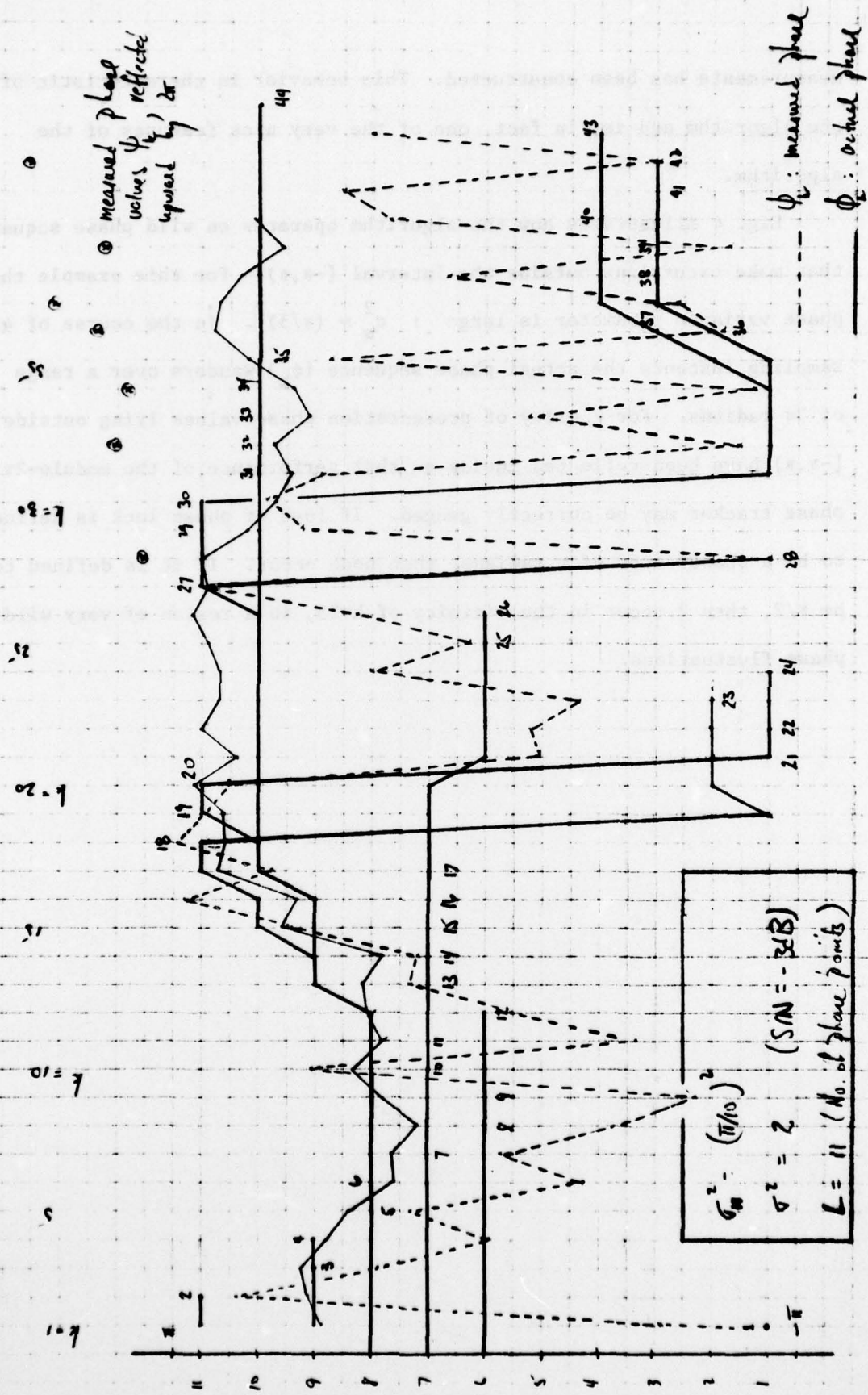
Fig. 3. Modulo-2π MAP Phase Sequences – S/N = –3dB

measurements has been constructed. This behavior is characteristic of the algorithm and is, in fact, one of the very nice features of the algorithm.

Fig. 4 illustrates how the algorithm operates on wild phase sequences that make excursions outside the interval $[-\pi,\pi)$. For this example the phase variance parameter is large : $\sigma_w^2 = (\pi/5)^2$. In the course of 45 sampling instants the actual phase sequence $\{\phi_k\}$ wanders over a range of 3π radians. For clarity of presentation phase values lying outside $[-\pi,\pi)$ have been reflected inside so that performance of the modulo-2π phase tracker may be correctly gauged. If loss of phase lock is defined to be a phase error of π radians, then none occur. If it is defined to be $\pi/2$, then 2 occur in the vicinity of k=25, in a region of very wild phase fluctuations.
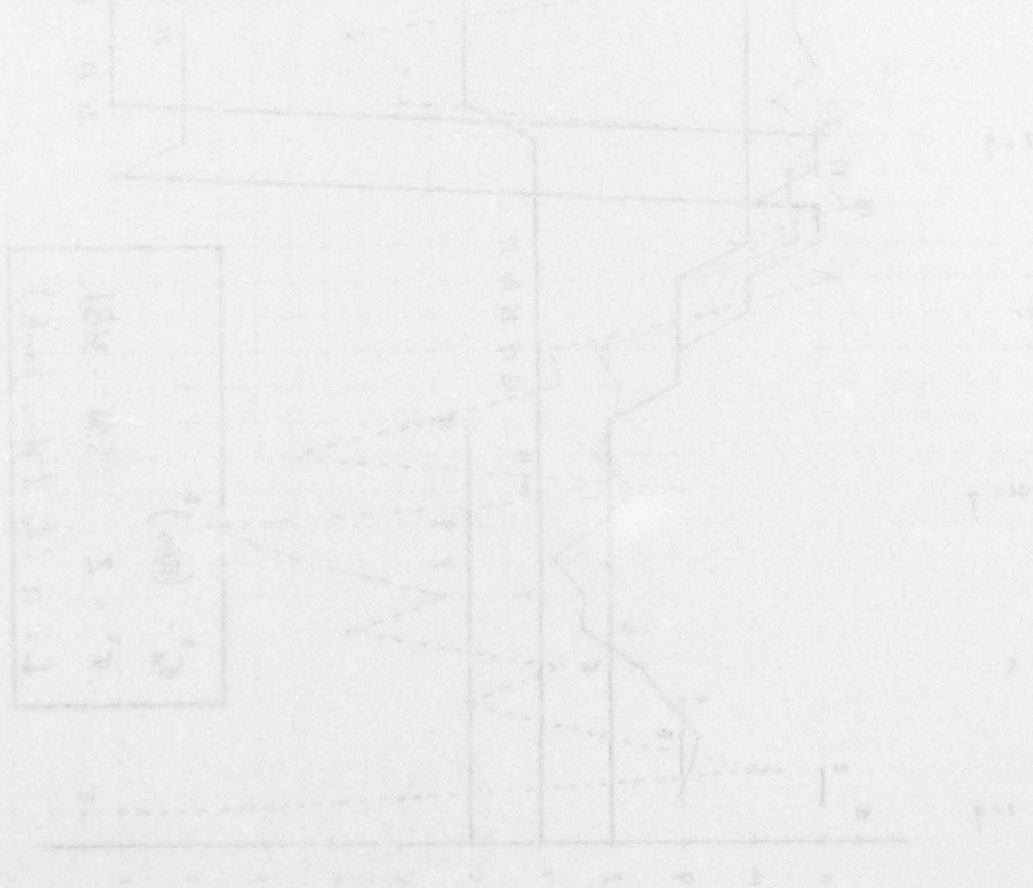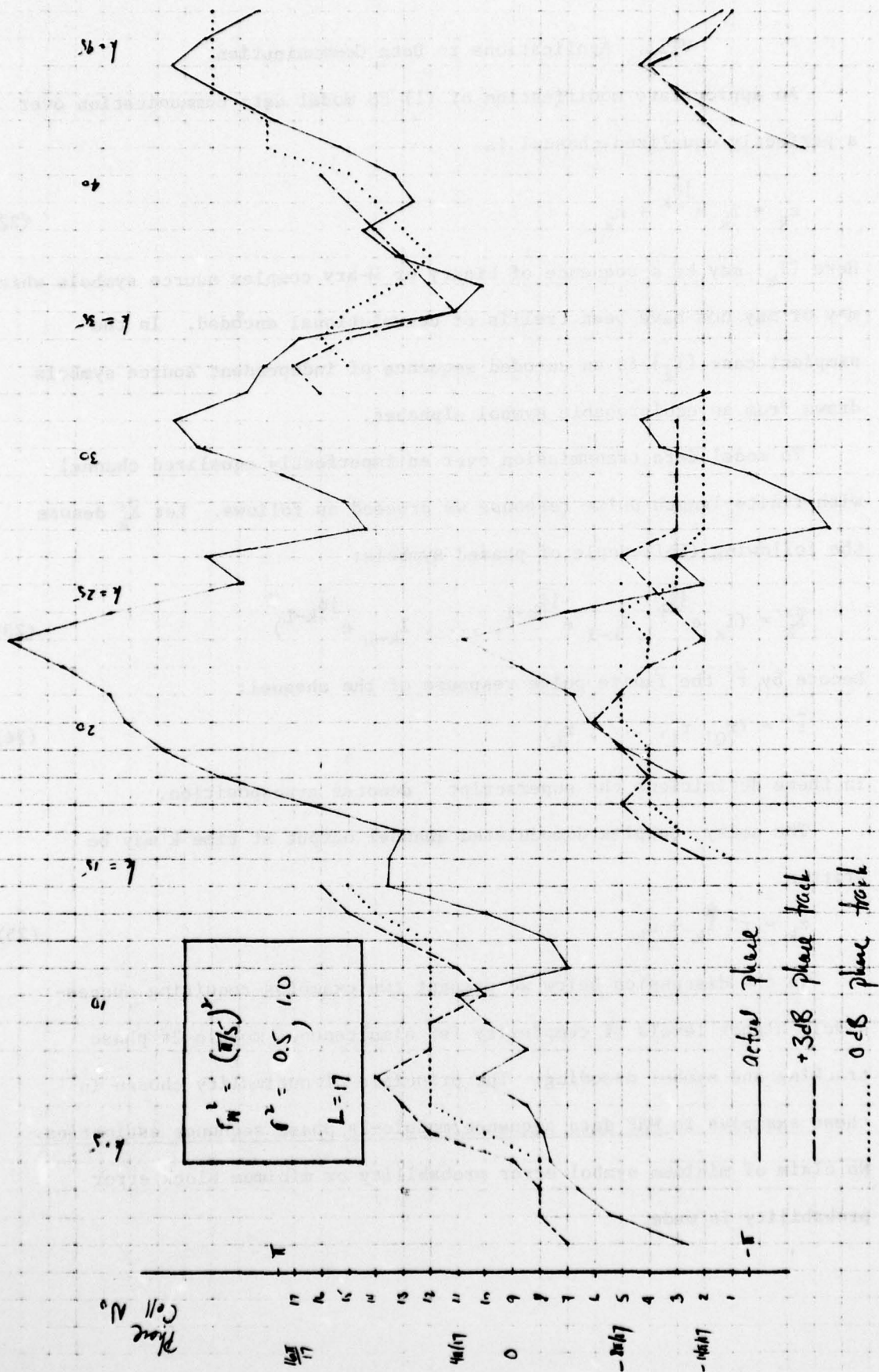
Fig. 4. MAP Phase Sequence for Wild Phase and Low Signal-Noise Ratio

VIII. Applications to Data Communication

An appropriate modification of (1) to model data communication over a perfectly equalized channel is

$$z_k = I_k \, e^{j\bar{\phi}_k} + n_k \tag{22}$$

Here $\{I_k\}$ may be a sequence of binary or M-ary complex source symbols which may or may not have been trellis or convolutional encoded. In the simplest case $\{I_k\}$ is an uncoded sequence of independent source symbols drawn from an equiprobable symbol alphabet.

To model data transmission over an imperfectly equalized channel with finite-length pulse response we proceed as follows. Let $\bar{X}_k'$ denote the following $(L+1)$-tuple of phased symbols:

$$\bar{X}_k' = (I_k \, e^{j\bar{\phi}_k}, \; I_{k-1} \, e^{j\bar{\phi}_{k-1}}, \; \ldots , \; I_{k-L} \, e^{j\bar{\phi}_{k-L}}) \tag{23}$$

Denote by $\bar{r}'$ the finite pulse response of the channel:

$$\bar{r}' = (r_0, \; r_1, \; \ldots , \; r_L) \tag{24}$$

In these definitions the superscript ´ denotes transposition.

The noisy, complex demodulated channel output at time k may be written

$$z_k = \bar{r}' \, \bar{X}_k + n_k \tag{25}$$

In the discussion below we present two examples requiring successively higher levels of complexity for simultaneous modulo-2π phase tracking and symbol decoding. The principle of optimality chosen in these examples is <u>MAP data sequence/modulo-2π phase sequence estimation.</u> No claim of minimum symbol error probability or minimum block error probability is made.

Example 1: $\{I_k\}$ is a sequence of independent symbols drawn from

an M symbol complex alphabet and transmitted over a perfectly equalized channel.

This example includes such modulation schemes as M-ary ASK, PSK, QASK, etc.

The number of candidate message sequences after K symboling periods is M^K.

The appropriate log density function for this example is[2]

$$\log f(\{z_k\}_1^K, \{\bar{\phi}_k\}_1^K, \{I_k\}_1^K)$$

$$= \log f(\{z_k\}_1^K / \{\bar{\phi}_k\}_1^K, \{I_k\}_1^K) + \log f(\{\bar{\phi}_k\}_1^K) \tag{26}$$

$$= C - \frac{1}{2\sigma^2} \sum_{k=1}^K |z_k - I_k e^{j\bar{\phi}_k}|^2 + \sum_{k=1}^K \log f(\bar{\phi}_k / \bar{\phi}_{k-1})$$

It follows easily that the path metric is

$$-\frac{1}{2\sigma^2} |z_k - I_k e^{j\bar{\phi}_k}|^2 + \log f(\bar{\phi}_k / \bar{\phi}_{k-1}) \tag{27}$$

This may be written (ignoring terms independent of $\bar{\phi}_k$, I_k)

$$\frac{1}{2\sigma^2} 2\text{Re } z_k I_k^* e^{-j\bar{\phi}_k} - \frac{1}{2\sigma^2} |I_k|^2 + \log f(\bar{\phi}_k / \bar{\phi}_{k-1}) \tag{28}$$

An even more explicit form of the path metric is

$$\frac{1}{2\sigma^2} 2C_k |I_k| \cos(\psi_k - \bar{\phi}_k - \theta_k) - \frac{1}{2\sigma^2} |I_k|^2 + \log f(\bar{\phi}_k / \bar{\phi}_{k-1}) \tag{29}$$

where (C_k, ψ_k) are the envelope and phase statistics defined previously and $|I_k|$ and θ_k are the magnitude and phase (usually visualized in a complex signal constellation) of the complex symbol I_k. From all three of the forms for the path metric it is clear that one is attempting to reconstruct phase-corrected symbols $I_k^* e^{-j\bar{\phi}_k}$ that correlate well with the received data while also satisfying a smoothness constraint (as embodied in the $f(\bar{\phi}_k / \bar{\phi}_{k-1})$) on the mod-$2\pi$ phase sequence.

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[2]This is actually a mixed density-probability mass function. However, because the probability mass function for $\{I_k\}_1^K$ assigns uniform mass M^{-K} to each K-tuple $\{I_k\}_1^K$, we ignore this term and call $f(\cdot, \cdot, \cdot)$ a density.

The appropriate trellis for Viterbi decoding of the modulo-2π phase sequence *and* the symbol sequence is the MxM symbol- phase trellis of pairs $(I^m, §_\ell)$, m=1,2,...,M, ℓ=1,2,...,M, corresponding to all allowable pairs of symbol and phase. Recall M denotes the number of phase points in the discretized modulo-2π phase space. If I_k is drawn from a binary alphabet for all k, then the symbol-phase trellis simply consists of 2M symbol-phase pairs over which survivor sequences are formed as outlined in Section VI.

Example 2: {I_k} a sequence of independent symbols drawn from an M symbol complex alphabet and transmitted over a channel with finite pulse response {r_ℓ}$_0^L$. For this example the appropriate log-density for the K measurements {z_k}$_1^K$ is (neglecting uninteresting constants)

$$\log f(\{z_k\}_1^K, \{\bar{\phi}_k\}_1^K, \{I_k\}_1^K) = \tag{30}$$

$$-\frac{1}{2\sigma^2} \sum_{k=1}^{K} |z_k - \bar{r}'\bar{X}_k|^2 + \sum_{k=1}^{K} \log f(\bar{\phi}_k/\bar{\phi}_{k-1})$$

Recall $\bar{X}_k$ depends on a specific set of source symbols and phases as in (27). If the modulo 2π phase space $[-\pi,\pi)$ is discretized to M values then there are M M^{L+1} distinct vectors $\bar{X}_k$.

The path metric for this problem is simply

$$-\frac{1}{2\sigma^2} |z_k - \bar{r}'\bar{X}_k|^2 + \log f(\bar{\phi}_k/\bar{\phi}_{k-1}) \tag{31}$$

The trellis for Viterbi decoding consists of the M M^{L+1} allowable $\bar{X}$-vectors corresponding to M phase values and M^{L+1} distinct symbol strings of length L+1. The path metric for each <u>allowable</u> transition in this trellis is computed as in (31) and used to construct survivor sequences. Of course, not all transitions are allowable because specification of

$(I_k, I_{k-1}, \cdots, I_{k-L})$ restricts the next allowable sequence to one of

M sequences beginning with I_{k+1} and terminating in I_k through I_{k-L+1}.

This means one must store $M \, M^{L+1}$ survivors ($=M \, 2^{L+1}$ for binary signalling)

and compute path metrics for $M \, M^{L+1} \, M \, M$ allowable transitions ($=M^2 \, 2^{L+2}$

for binary signalling).

IX. Conclusions

We have posed the problem of obtaining a MAP sequence estimate for

phase, modulo-2π. To solve the problem we have derived an expression

for transition probabilities in a modulo-2π version of random walk and

used the result to derive a dynamic programming algorithm for MAP sequence

estimation. Simulations indicate the performance of the algorithm is

excellent. Computational demands are modest, making the appraoch attrac-

tive for data communication applications.

Acknowledgments

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It is argued that only phase, modulo-2π, is of interest in data communication applications. The probabilistic evolution of random walk phase, modulo-2π, is studied and the results used to derive a MAP sequence estimator for phase on the interval [-π,π). The MAP sequence estimator is a Viterbi tracker (or dynamic programming algorithm) for tracking phase on a finite-dimensional trellis in [-π,π). Computational requirements are discussed and several representative simulations are included. Applications of the algorithm to

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carrier phase tracking in coherent data communication systems are outlined.